5.1 – Midsegment Theorem and Coordinate Proof

Midsegment of a triangle –

**THEOREM 5.1: MIDSEGMENT THEOREM**

The segment connecting the midpoints of two sides of a triangle is ______ to the third side and is _____ as long as that side.

\[ DE \parallel AC \text{ and } DE = \frac{1}{2} AC \]

**Example 1**  
*Use the Midsegment Theorem to find lengths*

Windows A large triangular window is segmented as shown. In the diagram, \( DF \) and \( EF \) are midsegments of \( \triangle ABC \). Find \( DF \) and \( AB \).

1. In Example 1, consider \( \triangle ADF \). What is the length of the midsegment opposite \( DF \)?

**Example 2**  
*Use the Midsegment Theorem*

In the diagram at the right, \( QS = SP \) and \( PT = TR \). Show that \( QR \parallel ST \).

**Solution**

Because \( QS = SP \) and \( PT = TR \), \( S \) is the ______ of \( QP \) and \( T \) is the ______ of \( PR \) by definition. Then \( ST \) is a ______ of \( \triangle PQR \) by definition and \( QR \parallel ST \) by the ______.

☑ **Checkpoint** Complete the following exercise.

2. In Example 2, if \( V \) is the midpoint of \( QR \), what do you know about \( SV \)?
5.2 – Use Perpendicular Bisectors

Perpendicular bisector –

Equidistant –

Concurrent –

Point of concurrency –

THEOREM 5.2: PERPENDICULAR BISECTOR THEOREM

In a plane, if a point is on the perpendicular bisector of a segment, then it is \( \text{__________} \) from the endpoints of the segment.

If \( CP \) is the \( \perp \) bisector of \( AB \), then \( CA = \_ \_ \_ \_ \_ \_ \_ \). 

THEOREM 5.3: CONVERSE OF THE PERPENDICULAR BISECTOR THEOREM

In a plane, if a point is equidistant from the endpoints of a segment, then it is on the \( \text{__________} \) \( \text{__________} \) of the segment.

If \( DA = DB \), then \( D \) lies on the \( \_ \_ \_ \_ \_ \_ \_ \) of \( AB \).

Example 1 Use the Perpendicular Bisector Theorem

\( AC \) is the perpendicular bisector of \( BD \). Find \( AD \).

Solution

\( AD = \_ \_ \_ \_ \_ \_ \_ \) \hspace{1cm} \text{Perpendicular Bisector Theorem}

\( \_ \_ \_ \_ \_ \_ \_ = \_ \_ \_ \_ \_ \_ \) \hspace{1cm} \text{Substitute.}

\( x = \_ \_ \_ \_ \_ \_ \) \hspace{1cm} \text{Solve for} \( x \).

\( AD = \_ \_ \_ \_ \_ \_ \_ = \_ \_ \_ \_ \_ \_ \_ = \_ \_ \_ \_ \_ \_ \_ \).
Example 2  Use perpendicular bisectors

In the diagram, \( KN \) is the perpendicular bisector of \( JL \).

a. What segment lengths in the diagram are equal?

b. Is \( M \) on \( KN \)?

Checkpoint In the diagram, \( JK \) is the perpendicular bisector of \( GH \).

1. What segment lengths are equal?

2. Find \( GH \).

THEOREM 5.4: CONCURRENCY OF PERPENDICULAR BISECTORS OF A TRIANGLE

The perpendicular bisectors of a triangle intersect at a point that is equidistant from the vertices of the triangle.

If \( PD \), \( PE \), and \( PF \) are perpendicular bisectors, then \( PA = \_\_\_\_ = \_\_\_\_\_\_ \).

Example 3  Use the concurrency of perpendicular bisectors

Football Three friends are playing catch. You want to join and position yourself so that you are the same distance from your friends. Find a location for you to stand.

Solution

Theorem 5.4 shows you that you can find a point equidistant from three points by using the

\( AD \) of the triangle formed by those points.

Copy the positions of points \( A \), \( B \), and \( C \) and connect those points to draw \( \triangle ABC \). Then use a ruler and a protractor to draw the three

\( AD \) of \( \triangle ABC \). The point of concurrency \( D \) is a location for you to stand.
5.3 – Use Angle Bisectors of Triangles

Incenter –

**THEOREM 5.5: ANGLE BISECTOR THEOREM**

If a point is on the bisector of an angle, then it is equidistant from the two ______ of the angle.

If \( AD \) bisects \( \angle BAC \) and \( DB \perp AB \) and \( DC \perp AC \), then \( DB = \) ______.

**THEOREM 5.6: CONVERSE OF THE ANGLE BISECTOR THEOREM**

If a point is in the interior of an angle and is equidistant from the sides of the angle, then it lies on the ______ of the angle.

If \( DB \perp AB \) and \( DC \perp AC \) and \( DB = DC \), then \( AD \) ______ \( \angle BAC \).

**Example 1** *Use the Angle Bisector Theorems*

Find the measure of \( \angle CBE \).

**Example 2** *Solve a real-world problem*

Web A spider’s position on its web relative to an approaching fly and the opposite sides of the web forms congruent angles, as shown. Will the spider have to move farther to reach a fly toward the right edge or the left edge?
Example 3  Use algebra to solve a problem

For what value of \( x \) does \( P \) lie on the bisector of \( \angle J \)?

\[
\begin{align*}
K & \quad x + 1 \\
P & \quad 2x - 5 \\
J & \quad L
\end{align*}
\]

THEOREM 5.7: CONCURRENCY OF ANGLE BISECTORS OF A TRIANGLE

The angle bisectors of a triangle intersect at a point that is equidistant from the sides of the triangle.

If \( AP \), \( BP \), and \( CP \) are angle bisectors of \( \triangle ABC \), then

\[
PD = \quad = \quad
\]

Example 4  Use the concurrency of angle bisectors

In the diagram, \( L \) is the incenter of \( \triangle FHJ \). Find \( LK \).

By the Concurrency of Angle Bisectors of a Triangle Theorem, the incenter \( L \) is \( \quad \) from the sides of \( \triangle FHJ \). So, to find \( LK \), you can find \( \quad \) in \( \triangle LHI \). Use the Pythagorean Theorem.

\[
\begin{align*}
\quad & = \quad \\
\quad & = \quad \\
\quad & = \quad \\
\quad & = \quad \\
\quad & = \quad \\
\quad & = \quad \text{Pythagorean Theorem} \\
\quad & = \quad \text{Substitute known values.} \\
\quad & = \quad \text{Simplify.} \\
\quad & = \quad \text{Take the positive square root of each side.}
\end{align*}
\]

Because \( \quad = LK \), \( LK = \quad \).
5.4 – Use Medians and Altitudes

**VOCABULARY**

Median of a triangle The median of a triangle is a segment from a vertex to the midpoint of the opposite side.

Centroid The point of concurrency of the three medians of a triangle is the centroid.

Altitude of a triangle An altitude of a triangle is the perpendicular segment from a vertex to the opposite side or to the line that contains the opposite side.

Orthocenter The point at which the lines containing the three altitudes of a triangle intersect is called the orthocenter of the triangle.

**THEOREM 5.8: CONCURREN CY OF MEDIANS OF A TRIANGLE**

The medians of a triangle intersect at a point that is two thirds of the distance from each vertex to the midpoint of the opposite side.

The medians of $\triangle ABC$ meet at $P$ and $AP = \frac{2}{3}$.

$BP = \frac{2}{3}$, and $CP = \frac{2}{3}$. 
Example 1  **Use the centroid of a triangle**  

In $\triangle FGH$, $M$ is the centroid and $GM = 6$. Find $ML$ and $GL$.

___ = ___ GL  

Concurrency of Medians of a Triangle Theorem

___ = ___ GL  

Substitute ___ for $GM$.

___ = ___ GL  

Multiply each side by the reciprocal, ___.

Then $ML = GL -$ ___ = ___ - ___ = ___.

So, $ML =$ ___ and $GL =$ ___.

**Checkpoint** Complete the following exercise.

1. In Example 1, suppose $FM = 10$. Find $MK$ and $FK$.

Example 2  **Find the centroid of a triangle**

The vertices of $\triangle JKL$ are $J(1, 2)$, $K(4, 6)$, and $L(7, 4)$. Find the coordinates of the centroid $P$ of $\triangle JKL$.

Sketch $\triangle JKL$. Then use the Midpoint Formula to find the midpoint $M$ of $JL$ and sketch median $KM$.

**THEOREM 5.9: CONCURRENCY OF ALTITUDES OF A TRIANGLE**

The lines containing the altitudes of a triangle are _______ .

The lines containing $AF$, $BE$, and $CD$ meet at $G$.

Example 3  **Find the orthocenter**

Find the orthocenter $P$ in the triangle.

a.  

b.
5.5 – Use Inequalities in a Triangle

**Example 1**  *Relate side length and angle measure*  
Mark the largest angle, longest side, smallest angle, and shortest side of the triangle shown at the right. What do you notice?

**Solution**

- largest angle
- longest side

The longest side and largest angle are _______ each other.

- shortest side
- smallest angle

The shortest side and smallest angle are _______ each other.

**THEOREM 5.10**

If one side of a triangle is longer than another side, then the angle opposite the longer side is _____ than the angle opposite the shorter side.

- $AB > BC$, so $\angle A > \angle C$.

**THEOREM 5.11**

If one angle of a triangle is larger than another angle, then the side opposite the larger angle is _____ than the side opposite the smaller angle.

- $\angle A > \angle C$, so $\overline{AB} > \overline{BC}$.

**Example 2**  *Find angle measures*  
Boating  A long-tailed boat leaves a dock and travels 2500 feet to a cave, 5000 feet to a beach, then 6000 feet back to the dock as shown below. One of the angles in the path is about 55° and one is about 24°. What is the angle measure of the path made at the cave?
Check point  Complete the following exercises.

1. List the sides of \( \triangle PQR \) in order from shortest to longest.

2. Another boat makes a trip whose path has sides of 1.5 miles, 2 miles, and 2.5 miles long and angles of 90°, about 53°, and about 37°. Sketch and label a diagram with the shortest side on the bottom and the right angle at the right.

Theorem 5.12: Triangle Inequality Theorem

The sum of the lengths of any two sides of a triangle is greater than the length of the third side.

\[
AB + AC > BC \\
AC + BC > AB \\
BC + AB > AC
\]

Example 3  Find possible side lengths

A triangle has one side of length 14 and another of length 10. Describe the possible lengths of the third side.

Solution

Let \( x \) represent the length of the third side. Draw diagrams to help visualize the small and large values of \( x \). Then use the Triangle Inequality Theorem to write and solve inequalities.
5.6 – Inequalities in Two Triangles and Indirect Proof

**THEOREM 5.13: HINGE THEOREM**

If two sides of one triangle are congruent to two sides of another triangle, and the included angle of the first is larger than the included angle of the second, then the third side of the first is ______ than the third side of the second.

**THEOREM 5.14: CONVERSE OF THE HINGE THEOREM**

If two sides of one triangle are congruent to two sides of another triangle, and the third side of the first is longer than the third side of the second, then the included angle of the first is ______ than the included angle of the second.

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**Example 1**

*Use the Converse of the Hinge Theorem*

Given that $AD \equiv BC$, how does $\angle 1$ compare to $\angle 2$?

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**Example 2**

*Solve a multi-step problem*

Travel Car A leaves a mall, heads due north for 5 mi and then turns due west for 3 mi. Car B leaves the same mall, heads due south for 5 mi and then turns $80^\circ$ toward east for 3 mi. Which car is farther from the mall?